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INFLATION, PRICES, AND INFORMATION IN COMPETITIVE SEARCH

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Abstract

Inflation, as a tax on money, induces buyers to reduce their money balances. Sellers are aware of this, so to attract costumers, they post price offers that reduce the need for buyers to carry precautionary money balances. We study this effect of inflation in a competitive search environment where buyers experience preference shocks after they are matched with a seller. With full information, equilibrium price offers consist of a flat fee which is independent of the quantities purchased. With private information of buyers' preferences, equilibrium price offers are restricted by incentive compatibility constraints. As a result, the price schedule that maps quantities purchased onto payments must be increasing. As inflation rises, these price schedules become relatively flat, so the marginal cost of purchasing goods is low. Consequently, buyers that are not liquidity constrained (with a low desire to consume) purchase inefficiently large quantities. Meanwhile, buyers with a high desire to consume typically purchase inefficiently low quantities because, as their money balances fall, they become liquidity constrained. This is in contrast with the full information benchmark where inflation reduces the quantities purchased by all buyers.

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1 Introduction

Many accounts stress that a commonly observed effect of high inflation is that individuals end up buying goods that have little value for them while they are liquidity constrained when they have a high desire to consume. For example, Willy Derkow, who was a student during the time of the German hyperinflation, remembered in 1975:¹ “As soon as you caught one (bundle of notes) you made a dash for the nearest shop and bought anything... You very often bought things you did not need.” With low inflation, this effect might not be so easily noticeable to a casual observer, but it is potentially an important adverse effect of inflation. In this paper, we advance a monetary search model where inflation reallocates goods from individuals with high valuations of goods to individuals with low valuations.

In our model, goods are traded in a competitive search market where money plays an essential role in facilitating transactions. This market combines trading frictions with an efficient mechanism for determining the terms of trade. The existence of trading frictions, such as lack of credit and private information, is needed not only to generate a role for money, but also to generate the cost of inflation we seek to capture. The efficiency of the mechanism for determining the terms of trade is desirable for our modeling purposes because it avoids that the distortions we seek to model are the result of an inferior trade mechanism between the private parties in a transaction.²

A key feature of our model is that buyers experience a preference shock after they have decided the demand for money and are already matched with a seller. This timing is important for our results. Firstly, it implies that buyers have an incentive to hold precautionary money balances because they face uncertain expenditure needs. As buyers reduce their precautionary balances to avoid the inflation tax, they are more likely to be liquidity constrained. Secondly, each seller serves a potential clientele of buyers who have different preferences *ex post*. This allows for the possibility of cross-subsidies across different buyer types. That is,

¹See www.johndclare.net/Weimar_hyperinflation.htm.

²As shown by Rocheteau and Wright (2005), with competitive search and full information the first best is attained at the Friedman rule, while this is not the case with Nash bargaining or Walrasian pricing and purely random search. See Kiyotaki and Wright (1989) for a seminal contribution on the search theoretic foundations of money.

the provision of inefficiently high quantities to buyers with low valuations and inefficiently low quantities to buyers with high valuations is a possible equilibrium outcome.

The main insights gained from the model are the following. Inflation gives buyers an incentive to reduce their money balances. Aware of this incentive, sellers try to attract buyers by posting price offers that reduce the amount of precautionary money balances that buyers need to carry. To this end, the posted price offers aim at reducing the variance of payments and thus the need for such precautionary balances. With full information, the equilibrium price offers consist of a flat fee which is independent of the quantity purchased by a buyer. As a result, buyers optimally choose an amount of money equal to the flat fee, so they avoid carrying any precautionary balances. With private information of preference shocks, sellers are forced to charge payments that increase with the quantities served due to incentive compatibility constraints. So a flat fee is not an equilibrium outcome. In this case, equilibrium price offers consist of an increasing non-linear price schedule. As inflation rises, price schedules become relatively flat as this reduces the variance of payments. With these flat price schedules, buyers choose to purchase inefficiently high quantities as long as they are not liquidity constrained (their desire to consume is low). Meanwhile, buyers with a high desire to consume often purchase inefficiently low quantities because they face binding liquidity constraints. Therefore, inflation reallocates output from buyers with a high desire to consume to buyers with a low desire to do so. A positive opportunity cost of holding money is crucial to this argument. If there is no opportunity cost to hold money balances, sellers post price schedules that reflect the marginal cost of production, so under the Friedman rule the first best is attained regardless of the privacy of information.³

The idea that inflation provides incentives to change trading arrangements in order to avoid idle or precautionary money balances is also found in two recent papers. In Faig and Huangfu (2004), inflation provides an incentive to market-makers to intermediate between buyers and sellers with the objective of eliminating idle money balances. In Berentsen, Camera, and Waller (2004), inflation provides a similar incentive to banks to do such intermediation. In our model, there is no intermediation between buyers and sellers from any

³The first best efficiency under the Friedman rule is not a robust feature of the model. In general, the revelation of private information creates welfare costs (see Faig and Jerez (2004)).

third party. Instead, it is the pricing mechanism that adjusts in order to reduce the need for idle money balances. Some of the consequences of inflation in our model are quite different from the earlier papers. In particular, the relocation of output from individuals with high valuations of output to individuals with low valuations is a novelty of our paper among this literature.

One of the major contributions of the paper is to introduce private information in monetary models with competitive search. To do so, we follow our treatment of private information in Faig and Jerez (2004) where we study a competitive search model of commerce in a non-monetary economy. The introduction of private information in either competitive search models or monetary models is a natural development which is gaining momentum. For example, Shimer and Wright (2004) recently advance a labor model of competitive search with private information. Meanwhile, Berentsen and Rocheteau (2004) and Ennis (2005) introduce private information in a monetary model, but without competitive search.

In a companion paper (Faig and Jerez, 2005), we argue that the precautionary demand for money explains well the dynamics of the historical velocity of circulation of money in the United States. The model in that paper simplifies the effect of inflation of the terms of trade, which we study here, by assuming a different timing of the preference shocks. There, preference shocks are realized after buyers decide their demand for money but prior to matching. As a result, sellers are able to post price offers that target particular buyer types. In competitive search equilibrium, buyers are then separated in different submarkets according to their type, so the possibility of cross-subsidization emphasized here is eliminated.

The structure of the paper is as follows. Section 2 describes the environment. Section 3 solves for the individuals' optimal financial decisions, including the demand for money. Sections 4 and 5 characterize the competitive search equilibrium with full and private information, respectively. Section 6 concludes. The proofs are gathered in the Appendix.

2 The Environment

There is a continuum of individuals with measure one who live in a large number of symmetric villages. The members of each village are ex ante identical. They all produce a perishable

good specific to their village and consume the goods produced in all villages except for their own. Hence, individuals must trade outside their village to consume.

Time is a discrete, infinite sequence of days. Each morning, an individual must choose to be either a buyer or a seller in the goods market that convenes later in the day. (One may think, for example, that buyers and sellers must perform distinct preparatory tasks. A similar choice is also present in Rocheteau and Wright, 2003, and Faig, 2004 with slightly different motivations.) Each day some of the members of the village will be buyers and others will be sellers. However, over time individuals will alternate between these two roles.

Individuals seek to maximize their expected lifetime utility:

$$E \sum_{t=0}^{\infty} \beta^t \mathcal{U}(\varepsilon, q_t^b, q_t^s), \quad (1)$$

where $\beta \in (0, 1)$ is the discount factor and

$$\mathcal{U}(\varepsilon, q^b, q^s) = \varepsilon U(q^b) - C(q^s) \quad (2)$$

is the one-period utility function. This function depends on the quantity consumed q^b if the individual chooses to be a buyer, and on the quantity produced q^s if he chooses to be a seller. It also depends on an idiosyncratic preference shock ε which affects the utility of consumption $\varepsilon U(q^b)$, but does not affect the disutility of production $C(q^s)$. The preference shock is uniformly distributed in the interval $[1, \bar{\varepsilon}]$, independent across time, and drawn in such a way that the Law of Large Numbers holds across individuals. The cumulative distribution function is then

$$F(\varepsilon) = \varphi(\varepsilon - 1), \quad (3)$$

where φ represents the constant density:⁴

$$\varphi = \frac{1}{\bar{\varepsilon} - 1}. \quad (4)$$

Both U and C are continuously differentiable and increasing. Also, U is strictly concave and C is convex, with $U(0) = C(0) = 0$, and $U'(0) = \infty$. Finally, there is a maximum quantity q^{\max} that the individual can produce each day which satisfies $\bar{\varepsilon} U(q^{\max}) \leq C(q^{\max})$.

⁴We assume a uniform distribution because it allows us to provide a simple characterization of a competitive search equilibria with private information. However, our main results should hold with a more general (non-degenerate) distribution function.

Money is an intrinsically useless, perfectly divisible, and storable asset. Units of money are called dollars. The supply of money grows at a constant factor γ , so

$$M_{+1} = \gamma M, \quad (5)$$

where M is the quantity of money per individual.⁵ Each day new money is injected via a lump-sum transfer τ common to all individuals:

$$\tau = (\gamma - 1) M. \quad (6)$$

We assume throughout the paper $\gamma > \beta$.

Each day goods are traded in a competitive search market, as in Moen (1997) and Shimer (1996). Prior to the search process, each seller simultaneously posts an offer which specifies the terms at which they commit to trade. Buyers then observe all the posted offers and direct their search towards the sellers posting the most attractive offer. The set of sellers posting the same offer and the set of buyers directing their search towards them form a submarket. In each submarket buyers and sellers from different villages meet randomly. To focus on the pricing issues we are interested in and avoid unnecessary complications, we assume a simple matching technology. We assume that individuals experience at most one match each period and that matching is efficient. That is, individuals can avoid matching with fellow villagers and the short-side of the market is always served. As a result, the probability that a buyer meets a suitable seller in a submarket is

$$\pi^b(\alpha) = \min(1, \alpha), \quad (7)$$

where α is the ratio of sellers over buyers in that submarket. Similarly, the probability that a seller meets a suitable buyer is

$$\pi^s(\alpha) = \min(1, \alpha^{-1}). \quad (8)$$

When a buyer and a seller meet in a submarket they trade according to the pre-specified offer.

⁵For simplicity, the subscript t is omitted in most expressions of the paper, so, for example, M stands for M_t and M_{+1} stands for M_{t+1} .

In the competitive search market individuals are anonymous and enforcement is limited. This implies that money is essential due to the absence a double coincidence of wants (generated by the ex-ante choice of trading roles). However, inside each village financial contracts are enforceable. In particular, in each village there is a competitive credit market where a one-period risk-free bond is traded and a competitive insurance market where individuals can insure against their preference shocks. As it will become apparent, these two centralized markets exhaust the gains from trade inside the village.

The village structure we adopt in this paper allows for a coherent coexistence of money and financial assets. Moreover, the ability of individuals to rebalance their portfolio in their village renders a tractable distribution of money balances. As discussed in Faig (2004), this role is similar to the roles played by large households in Shi (1997) and centralized markets for goods in Lagos and Wright (2005). We adopt the village structure because it proves the most useful for our analysis.⁶

A typical day proceeds as follows (see Table 1). In the morning, centralized financial markets are open in each village. During this time, financial contracts from the previous day are settled. The government hands out monetary transfers that increase the money supply. Individuals decide whether to be buyers or sellers. Then they adjust their holdings of bonds and money, and purchase insurance if they wish. At noon financial markets close and the goods market opens. As a result of the competitive search process submarkets are formed. When a buyer and a seller meet in a submarket, the buyer experiences the preference shock ε and the agents trade according to the pre-specified offer. As a result of trade, sellers produce, buyers consume, and money changes hands from buyers to sellers.

Table 1

⁶With the village structure we avoid having the double decision layer of large representative households. Moreover, we do not need the existence of goods traded in centralized markets and quasi-linear preferences.

MORNING			AFTERNOON		
Financial markets are open			Goods market is open		
Previous financial claims settled	Choice buyer-seller. Sellers post offers	Choice of bonds, money, insurance	Buyers choose among offers	Realization preference shock	Trades takes place

Our equilibrium concept combines perfect competition in financial markets with competitive search in the goods market. In equilibrium, individuals make optimal choices taking as given the environment where they live. This environment includes a sequence of nominal interest rates and insurance premia, and a sequence of conditions in the goods market to be detailed below (essentially the reservation surpluses of other traders). Individuals also have rational expectations about the future conditions of this environment. We focus on symmetric and stationary equilibria where all individuals use identical strategies and real allocations are constant over time.

To characterize an equilibrium, we proceed as follows. First, we describe the buyer-seller occupational choice and the financial decisions of a representative individual given some conjectures about the equilibrium nominal interest rates and insurance premia, as well as the conditions in the goods market. Then we characterize the conditions in the goods market in a competitive search equilibrium and show that they satisfy our former conjecture. Finally, we check that the conjectured interest rate and insurance premia clear the financial markets. A formal definition of an equilibrium is provided at the end of Section 4.

3 Buyer-Seller Choice and Financial Decisions

Consider an individual facing the following environment:

In the credit market, a one-period risk-free bond is traded. The nominal interest rate is:

$$i = \frac{\gamma - \beta}{\beta}. \quad (9)$$

Since good prices are proportional to M , which grows at the factor γ , the real interest rate is then equal to the subjective discount rate: $\beta^{-1} - 1$.

In the insurance market, an individual that decides to be a buyer can purchase an insurance contract which delivers a certain dollar amount μ_ε^b next day contingent on experiencing a shock ε in the afternoon. The premium $\tilde{\mu}^b$ of such a contract is actuarially fair:

$$\tilde{\mu}^b = \int_1^\varepsilon \mu_\varepsilon^b dF(\varepsilon). \quad (10)$$

While individuals potentially face a risk of meeting a trading partner or not, in our environment there is no need for insuring against such risks because they vanish in equilibrium (all individuals trade with probability one).

We make the conjecture that the goods market has a unique active submarket in equilibrium where all individuals trade. Let α be the ratio of buyers over sellers in the submarket. Also, let $\{q_\varepsilon, d_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ be the terms of trade contingent of the realization of ε (the buyer's type) where q_ε denotes the quantity purchased and d_ε is the total payment in dollars of a type- ε buyer. Since payments change over time as the money supply grows, the terms of trade may also be described by $\{q_\varepsilon, z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ where

$$z_\varepsilon = \frac{\beta d_\varepsilon}{M_{+1}} \quad (11)$$

is the payment of a type- ε buyer measured in utils. In a stationary equilibrium the pairs $(q_\varepsilon, z_\varepsilon)$ are time invariant.

Prior to all financial choices, each morning the individual chooses the trading role that yields maximal utility. The value function V of the individual at the beginning of a day then obeys:

$$V\left(\frac{A}{M}\right) = \max\left\{V^b\left(\frac{A}{M}\right), V^s\left(\frac{A}{M}\right)\right\}; \quad (12)$$

where A is the initial wealth in dollars, and V^b and V^s are the value functions conditional on being a buyer or a seller during the day, respectively. The money supply is used to deflate nominal quantities. This deflator is appropriate because nominal prices increase proportionately with M (see (5) and (6)). The ratio A/M can be interpreted as initial real wealth and is denoted by a .

While financial markets are open, the individual reallocates wealth and may also purchase insurance. Conditional on being a buyer the individual chooses the demands for money and

bonds, m^b and b^b , and the insurance contract, $\{\mu_\varepsilon^b\}_{\varepsilon \in [1, \bar{\varepsilon}]}$, to solve:

$$V^b(a) = \max_{m^b, b^b, \{\mu_\varepsilon^b\}_{\varepsilon \in [1, \bar{\varepsilon}]}} \int_1^{\bar{\varepsilon}} \left\{ \pi^b(\alpha) [\varepsilon U(q_\varepsilon) + \beta V(a_{+1}^{b\varepsilon})] + [1 - \pi^b(\alpha)] \beta V(a_{+1}^{b0}) \right\} dF(\varepsilon) \quad (13)$$

subject to

$$a_{+1}^{b\varepsilon} = \frac{m^b + b^b(1+i) + \mu_\varepsilon^b - \tilde{\mu}^b + \tau - d_\varepsilon}{M_{+1}}, \quad (14)$$

$$a_{+1}^{b0} = \frac{m^b + b^b(1+i) - \tilde{\mu}^b + \tau}{M_{+1}}, \quad (15)$$

$$a = \frac{m^b + b^b}{M}, \text{ and} \quad (16)$$

$$m^b \geq d_\varepsilon \text{ for all } \varepsilon \in [1, \bar{\varepsilon}]. \quad (17)$$

The buyer meets a seller with probability $\pi^b(\alpha)$. Conditional on the realization of the preference shock ε , the buyer purchases q_ε for d_ε dollars, so next period's real wealth $a_{+1}^{b\varepsilon}$ is given by (14). If the buyer does not meet a seller, she buys nothing and next period's real wealth a_{+1}^{b0} is given by (15). The choice of how to allocate wealth between money m^b and bonds b^b must satisfy the budget constraint (16). The buyer must also carry enough money to face all contingent payments, so m^b must satisfy (17).

Conditional on being a seller the individual chooses the demands for money m^s and bonds b^s to solve:

$$V^s(a) = \max_{m^s, b^s} \int_1^{\bar{\varepsilon}} \left\{ \pi^s(\alpha) [\beta V(a_{+1}^{s\varepsilon}) - C(q_\varepsilon)] + [1 - \pi^s(\alpha)] \beta V(a_{+1}^{s0}) \right\} dF(\varepsilon) \quad (18)$$

subject to

$$a_{+1}^{s\varepsilon} = \frac{m^s + b^s(1+i) + \tau + d_\varepsilon}{M_{+1}}, \quad (19)$$

$$a_{+1}^{s0} = \frac{m^s + b^s(1+i) + \tau}{M_{+1}}, \quad (20)$$

$$a = \frac{m^s + b^s}{M}, \text{ and} \quad (21)$$

$$m^s \geq 0. \quad (22)$$

The seller meets a buyer with probability $\pi^s(\alpha)$ and, contingent on the buyer's type, sells q_ε for d_ε dollars. If the seller does not meet a buyer he sells nothing. Next period real wealth

in each event is given by (19) and (20). The budget constraint (21) must be satisfied and money cannot be negative, (22).

In addition to all constraints specified above, the individual faces an endogenous lower bound on next period real wealth because he or she must be able to repay the amounts borrowed with probability one without reliance to unbounded borrowing (No-Ponzi game condition):

$$a_{+1} \geq a_{\min} \text{ with probability one.} \quad (23)$$

Here a_{+1} denotes the stochastic real wealth next period, which depends on the occupational choice, the realization of ε , and the outcome of the trading match. The endogenous lower bound a_{\min} is equal to minus the present discounted value of the maximum guaranteed income the individual can obtain as a seller.

The optimization program described in equations (12) to (23) is easily solved once the value function V is known. The value function V is a well defined function of a that can be characterized using standard recursive methods. The following result shows that V is concave with a linear segment (see Appendix A for the proof).

Proposition 3.1 *There is an interval $[\underline{a}, \bar{a}] \subset [a_{\min}, \infty)$ where the equilibrium value function V takes the linear form*

$$V(a) = v_0 + a \quad (24)$$

with v_0 independent of a . Outside this interval, V is strictly concave and continuously differentiable. Also, the interval $[\underline{a}, \bar{a}]$ is absorbing, that is $a \in [\underline{a}, \bar{a}]$ implies $a_{+1} \in [\underline{a}, \bar{a}]$ with probability one.

The linear segment of V is due to the endogenous choice of trading roles individuals make each day. Intuitively, if an individual is not rich enough to afford being a buyer forever and not so poor to have to be a seller every day, then the individual will alternate between being the two trading roles. As the individual does so, wealth does not affect the quantities consumed or produced, instead it affects how often and how early the individual consumes or produces. Since utility is linear on the times and the timing an individual consumes and produces, the value function is linear.

The property that the interval $[\underline{a}, \bar{a}]$ is absorbing simplifies the model dramatically. If all individuals have initial wealth in the interval $[\underline{a}, \bar{a}]$, as we assume from now on, the behavior of buyers and sellers is independent from their wealth. As we show in the next section, this implies that there is no incentive to create submarkets that cater to individuals of different wealth. In this case, the distribution of money holdings is easily characterized.

The optimal demands for money are easily derived given that money earns not interest but bonds earn $i > 0$. This implies that it is not optimal to carry money balances that are never used. Therefore, m^b is equal to the highest contingent payment: $m^b = \max \{d_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ and $m^s = 0$. Using these optimal demands for money, (24), and $a_{+1} \in [\underline{a}, \bar{a}]$ with probability one, the value functions of the buyer (13) and the seller (18) simplify into:

$$V^b(a) = S^b + \beta \left(v_0 + \frac{\gamma - 1}{\gamma} \right) + a \text{ and} \quad (25)$$

$$V^s(a) = S^s + \beta \left(v_0 + \frac{\gamma - 1}{\gamma} \right) + a. \quad (26)$$

These value functions differ only in the first term. This term represents the expected trading surpluses of buyers and sellers in the afternoon goods market:

$$S^b \equiv \int_1^{\bar{\varepsilon}} \pi^b(\alpha_\varepsilon) [\varepsilon U(q_\varepsilon) - z_\varepsilon] dF(\varepsilon) - im, \text{ and} \quad (27)$$

$$S^s \equiv \int_1^{\bar{\varepsilon}} \pi^s(\alpha_\varepsilon) [z_\varepsilon - C(q_\varepsilon)] dF(\varepsilon). \quad (28)$$

In (27), m denotes real money balances in utils. Since buyers carry an amount of money equal to the highest contingent payment, we have

$$m \equiv \beta m^b / M_{+1} = \max \{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]} . \quad (29)$$

Note that the insurance coverages are missing from (27). As long as $a \in [\underline{a}, \bar{a}]$ the buyer is indifferent between purchasing insurance or not. The only role played by insurance markets is to ensure that wealth does not drift out of the interval $[\underline{a}, \bar{a}]$. This role is only important if buyers purchase nothing for low realizations of ε . If buyers purchase positive amounts for all realizations of ε then, in general, insurance markets are redundant. In this case, the individual prevents a_{+1} from drifting below \underline{a} by choosing to be a seller and prevents a_{+1} from drifting above \bar{a} by choosing to be a buyer.⁷

⁷See the Appendix for the details.

4 Competitive Search with Full Information

In this section we characterize a competitive search equilibrium in the goods market given the individual optimal financial decisions. We show that in equilibrium the goods market has the properties conjectured in Section 3. Moreover, the nominal interest rate in (9) and insurance premia in (10) clear the financial markets. Then we characterize a symmetric monetary stationary equilibrium where all individuals have initial wealth $a \in [a, \bar{a}]$.

Prior to matching and while buyers can still rebalance the quantity of money they hold sellers post their offers. An offer is a schedule $\{(q_\varepsilon, z_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}$, by means of which a seller commits to sell q_ε units of output in exchange of a real payment z_ε in the event of being matched with a buyer of type ε .⁸ All individuals have rational expectations regarding the number of buyers that will be attracted by each offer, and thus about the relative proportion of buyers and sellers in each submarket. The set of offers posted in equilibrium must be such that sellers have no incentives to post deviating offers.

A submarket is characterized by an offer and a ratio of buyers over sellers, $[\alpha, \{(q_\varepsilon, z_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}]$. Let Ω be the set of all submarkets that are formed in equilibrium. A competitive search equilibrium is a set $\{\Omega, \bar{S}^b, \bar{S}^s\}$ such that

1. All buyers attain the same expected surplus \bar{S}^b .
2. All sellers attain the same expected surplus \bar{S}^s .
3. The expected surpluses of buyers and sellers are identical: $\bar{S}^b = \bar{S}^s$.
4. Each $\omega \in \Omega$ solves the following program:

$$\bar{S}^b = \max_{[\alpha, \{(q_\varepsilon, z_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}} \int_1^{\bar{\varepsilon}} \{\pi^b(\alpha) [\varepsilon U(q_\varepsilon) - z_\varepsilon]\} dF(\varepsilon) - im \quad (30)$$

⁸We could allow for offers which are contingent both on the type ε and the wealth a of the buyer. However, since the buyers' expected surplus (27) and money balances (29) are independent of a , all buyers of a given type ε are identical from the sellers' view point. Hence, restricting to offers which are only contingent on ε is without loss of generality.

subject to

$$\max \{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]} = m, \quad (31)$$

$$\int_1^{\bar{\varepsilon}} \{\pi^s(\alpha) [z_\varepsilon - C(q_\varepsilon)]\} dF(\varepsilon) = \bar{S}^s, \text{ and} \quad (32)$$

$$\alpha, q_\varepsilon \geq 0 \quad (33)$$

Conditions 1 to 3 are straightforward. Buyers are free to choose the submarket where they trade and they have identical payoff functions (27) in the relevant interval of wealth, so they must attain the same expected surplus. The same is true for sellers. Moreover, for trade to occur in equilibrium there must be buyers and sellers present in that submarket, so individuals must be indifferent between the two trading roles. Condition of equilibrium 4 results from a combination of optimal behavior and competition among sellers when they post their price offers. In words, this condition says that buyers choose among submarkets in order to maximize their expected surplus subject to their cash constraint and the constraint that sellers receive a common expected surplus \bar{S}^s . (Individuals are infinitesimal in the market, so they take as given the expected surplus of other individuals.) The cash constraint (31) ensures that the buyer is able to pay for the good in all possible contingencies. Constraint (32) is an arbitrage condition. If a seller tries to post an offer that attracts buyers and yields a higher expected surplus, other sellers would profitably undercut this offer (e.g. by offering those buyers the same quantity for a slightly lower payment). Sellers never post deviating offers that imply a lower expected surplus because they can attain \bar{S}^s in the current submarket.

The price offers solving (30) to (33) are not restricted to ensure that the trade surplus of the buyer is always positive ex-post. Such ex-post rationality constraint would be natural if buyers and sellers could not communicate prior to the realization of the preference shock. However, we assume that buyers are able to commit to trade according to the offer posted in the submarket they visit by making a down payment equal to $\min \{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ before ε is realized. Such a commitment allows for the strong analytical results we report in the paper. In its absence, the equations characterizing an equilibrium are easy to state (see Statement 19 in Appendix B) but difficult to analyze without relying on numerical methods.

A solution for program (30) to (33) must have the following two characteristics. Firstly,

buyers and sellers must trade with probability one in any active submarket:

$$\alpha = \pi^b(\alpha) = \pi^s(\alpha) = 1. \quad (34)$$

Condition (34) is necessary for the total expected surplus from a match to be maximal subject to the cash constraint. Secondly, the payments from the buyer to the seller must be uniform:

$$z_\varepsilon = m \text{ for } \varepsilon \in [1, \bar{\varepsilon}]. \quad (35)$$

To see this, notice that the sellers' expected surplus (32) depends on the buyer's average payment, but it does not depend on higher moments of the distribution of $\{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$. In contrast, for a given average payment, a buyer prefers a smooth distribution of $\{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ because the opportunity cost of holding money depends on the maximum payment. Therefore, all solutions to (30) to (33) must satisfy (35).

Substituting (35) and (34) into (32) yields

$$m = \bar{S}^s + \int_1^{\bar{\varepsilon}} C(q_\varepsilon) dF(\varepsilon). \quad (36)$$

Using (35) to (36), program (30) to (33) simplifies into

$$\bar{S}^b = \max_{\{q_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}} \int_1^{\bar{\varepsilon}} [\varepsilon U(q_\varepsilon) - (1+i)C(q_\varepsilon)] dF(\varepsilon) - (1+i)\bar{S}^s. \quad (37)$$

The equilibrium quantities that solve this program are given by the following first order condition:

$$\varepsilon U'(q_\varepsilon) = (1+i)C'(q_\varepsilon) \text{ for } \varepsilon \in [1, \bar{\varepsilon}]. \quad (38)$$

With full information, the inflation tax (positive i) creates a proportional wedge $(1+i)$ between the marginal utility of consumption and the marginal cost of production in the same fashion as in economies with a cash-in-advance constraint.

To complete the characterization of a competitive search equilibrium, it remains is to determine \bar{S}^s . Since buyers and sellers attain the same expected surplus, (36) and (37) imply:

$$m = \frac{1}{2+i} \int_1^{\bar{\varepsilon}} [\varepsilon U(q_\varepsilon) + C(q_\varepsilon)] dF(\varepsilon). \quad (39)$$

We are ready to define an equilibrium of the monetary economy:

A **monetary stationary equilibrium** is a vector of real numbers $(i, \alpha, m, \bar{S}^s)$ and a set of real functions $\{(q_\varepsilon, z_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ that satisfy the system of equations: (9), (34), (35), (36), (38), and (39).

This equilibrium is consistent with the environment conjectured in Section 3. In particular, since the solution to program (30) to (31) is unique, there is at most one active submarket. The credit market clears because individuals have a perfectly elastic net demand for bonds at the interest rate (9). The insurance market clears because insurance premia are fair. These financial markets exhaust the gains for trading financial securities inside the village because all individuals have identical marginal rates of substitution in their margins of choice made in the morning.

We have shown that equilibrium offers minimize the opportunity cost of money balances by having z_ε identical for all ε . Buyers optimally choose an amount of money m equal to the uniform payment and spend all their cash. The welfare effects of inflation are captured by equations (36), (38) and (39), together with the equation that determines the equilibrium nominal interest rate (9). At the Friedman rule, $i \rightarrow 0$, the quantities of output traded are efficient. The convexity of C and concavity of U imply that q_ε is an increasing function of ε , so high types purchase more output than low types. As inflation rises the opportunity cost of holding money increases inducing buyers to reduce their money holdings. Sellers adjust their offers by reducing their fees and the quantities handed to each buyer type. That is, q_ε is a decreasing function of i for all ε . These reductions of output relative to the efficient quantities represent the welfare cost of inflation in this model.

It is important to note that the equilibrium is only implementable if sellers can observe the realization of the preference shocks. With a uniform payment high types receive more output than low types and yet pay the same. Unless shocks are observed by the seller, buyers have clear incentives to lie and claim they have the highest type $\bar{\varepsilon}$. In the next section, we characterize the equilibrium when preference shocks are private information.

5 Competitive Search with Private Information

Consider the competitive search market described in the previous section, but with shocks that are privately observed by the buyers that experience them.⁹ In this case, the offers posted by sellers must be incentive compatible. That is, buyers must have no incentives to lie about their type. Program (30) to (33) is then further restricted to satisfy the incentive compatibility constraint:¹⁰

$$\varepsilon' \in \arg \max_{\varepsilon \in [1, \bar{\varepsilon}]} [\varepsilon' U(q_\varepsilon) - z_\varepsilon], \text{ for all } \varepsilon' \in [1, \bar{\varepsilon}] \quad (40)$$

As is standard, constraint (40) can be restated using the following well-known result. (See Mas-Colell, Winston and Green, 1995, Proposition 23.D.2.)

Lemma 5.1 *Let the indirect ex-post trade surplus of a type- ε buyer be defined as*

$$v_\varepsilon \equiv \varepsilon U(q_\varepsilon) - z_\varepsilon. \quad (41)$$

A trading offer satisfies the incentive compatibility constraint (40) if and only if q_ε is non-decreasing in ε and v_ε satisfies

$$v_\varepsilon - v_1 = \int_1^\varepsilon \frac{\partial}{\partial x} [xU(q_x) - z_x] dx = \int_1^\varepsilon U(q_x) dx, \text{ for all } \varepsilon \in [1, \bar{\varepsilon}]. \quad (42)$$

Using Lemma 5.1, (34), and (41), the maximization program (30) to (33) with the restriction (40) can be restated as the following optimal control problem:

⁹If shocks are not observable in the village of origin insurance may not exist. This is irrelevant for the characterization of an equilibrium as we define it because V is affine in the relevant segment. However, the absence of insurance changes the values of \underline{a} and \bar{a} in (58) and so the set of parameter values for which an equilibrium exists.

¹⁰Formally, an offer $\{(q_\varepsilon, z_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ is a direct revelation mechanism that is incentive compatible. While direct revelation mechanisms can in principle be random, this is only optimal provided absolute risk aversion decreases with ε (see Maskin and Riley (1984)). In our environment absolute risk aversion is independent of ε , so random mechanisms are never used in equilibrium. We therefore restrict to deterministic mechanisms. See, however, Shimer and Wright (2004) for a different environment with indivisibilities where random mechanisms are optimal.

$$\bar{S}^b = \max_{[m, \{(q_\varepsilon, v_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]})} \int_1^{\bar{\varepsilon}} v_\varepsilon dF(\varepsilon) - im \quad (43)$$

subject to

$$\int_1^{\bar{\varepsilon}} [\varepsilon U(q_\varepsilon) - C(q_\varepsilon) - v_\varepsilon] dF(\varepsilon) = \bar{S}^s, \quad (44)$$

$$\varepsilon U(q_\varepsilon) - v_\varepsilon \leq m \text{ for } \varepsilon \in [1, \bar{\varepsilon}], \quad (45)$$

$$\dot{v}_\varepsilon = U(q_\varepsilon) \text{ for } \varepsilon \in [1, \bar{\varepsilon}], \text{ and} \quad (46)$$

$$q_\varepsilon \text{ is non-decreasing in } \varepsilon, \quad (47)$$

$$m, q_\varepsilon \geq 0. \quad (48)$$

The control variable of this problem is q_ε while v_ε is the state variable. Using the Maximum Principle, the optimal path for the control variable q_ε must satisfy the following equation (see the Appendix B for the derivation) :

$$\begin{cases} (\varepsilon - \gamma_2) U'(q_\varepsilon) = \gamma_1 C'(q_\varepsilon) & \text{for } \varepsilon \in [1, \hat{\varepsilon}], \quad \text{and} \\ q_\varepsilon = q_{\hat{\varepsilon}} \equiv \hat{q} & \text{for } \varepsilon \in [\hat{\varepsilon}, \bar{\varepsilon}]; \end{cases} \quad (49)$$

where γ_1 , γ_2 , and $\hat{\varepsilon}$ are positive numbers given by:

$$\gamma_1 = \frac{1+i}{1+2i}, \quad (50)$$

$$\gamma_2 = \frac{i}{1+2i}, \text{ and} \quad (51)$$

$$\gamma_1 + \frac{\gamma_2}{\bar{\varepsilon}} = \frac{\hat{\varepsilon}}{\bar{\varepsilon}} + \frac{1}{2} \left[1 - \left(\frac{\hat{\varepsilon}}{\bar{\varepsilon}} \right)^2 \right]. \quad (52)$$

The variable $\hat{\varepsilon}$ represents the break-point shock where the cash constraint becomes binding. Buyers with a realization of the preference shock lower than $\hat{\varepsilon}$ keep some cash balances unspent. Buyers with shocks higher or equal to $\hat{\varepsilon}$ spend all their cash. Combining (50) to (52), we obtain $\hat{\varepsilon}$ as an implicit function of i :

$$\frac{i}{1+2i} \frac{\bar{\varepsilon}}{\varphi} = \frac{(\bar{\varepsilon} - \hat{\varepsilon})^2}{2}. \quad (53)$$

This equation implies that $\hat{\varepsilon}$ is a decreasing function of the nominal interest rate i . Intuitively, as i increases, individuals reduce the amount of money balances they wish to carry, so the probability of being liquidity constrained increases.

The other unknowns of program (43) to (48) are determined as follows. The optimal path for the state variable v_ε is implied by the differential equation (46) for a given initial value v_1 . The optimal value of m is given by (45) with equality at the break-point $\hat{\varepsilon}$. The value v_1 in equilibrium is determined by the condition $\bar{S}^s = \bar{S}^b$. That is, v_1 must be such that

$$\int_1^{\bar{\varepsilon}} v_\varepsilon dF(\varepsilon) - im = \int_1^{\bar{\varepsilon}} [\varepsilon U(q_\varepsilon) - C(q_\varepsilon) - v_\varepsilon] dF(\varepsilon). \quad (54)$$

Finally, the underlying payments $\{z_\varepsilon\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ are calculated from (41).

The formal definition of the equilibrium is now the following:

A **monetary stationary equilibrium** is a vector of real numbers $(i, \gamma_1, \gamma_2, \hat{\varepsilon}, \alpha, m, \bar{S}^s, \bar{S}^b)$ and a set of real functions $\{(q_\varepsilon, v_\varepsilon)\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ that satisfy the system of equations: (9), (34), (43), (44), (45) with equality at $\hat{\varepsilon}$, (46), (49), (50), (51), (53), and (54).

With private information, equilibrium payments do not consist of a flat fee. Instead, payments must be increasing with the quantity of output purchased to satisfy the incentive compatibility constraints generated by private information. The derivative of the payment z relative to the output q in a transaction can be calculated using the chain rule of differentiation together with (41), (46), and (49). The resulting expression is:

$$\frac{dz}{dq} = \gamma_1 C'(q) + \gamma_2 U'(q) \quad \text{for } \varepsilon \in (1, \hat{\varepsilon}). \quad (55)$$

Since both γ_1 and γ_2 are positive, the derivative (55) is positive. That is, the price schedule that maps the quantities of output purchased with the corresponding payments is upward sloping. Furthermore, using (49), (50), (51), we can calculate how the slope of this implicit price schedule changes with the nominal interest rate:

$$\frac{d}{di} \frac{dz}{dq} = \frac{1}{1 + 2i} \frac{1 - \varepsilon}{\varepsilon - \gamma_2} C'(q) < 0 \quad \text{for } \varepsilon \in (1, \hat{\varepsilon}) \quad (56)$$

As the nominal interest i increases, the cost of carrying idle money balances rises. Aware of this, sellers have an incentive to post price offers that imply a lower variability of payments, which is equivalent to a lower slope of the implicit price schedules that map quantities of output with payments. Associated with this flatter price schedules, the quantities of output

purchased as long as $\varepsilon \in (1, \hat{\varepsilon})$ increase with the nominal interest rate i . This relationship results from the application of the Implicit Function Theorem to the system of equations (49) to (51) which yields:

$$\frac{dq_\varepsilon}{di} = \frac{\varepsilon - 1}{(1+i)(1+2i)} \frac{U'(q_\varepsilon)}{\gamma_1 C''(q_\varepsilon) - (\varepsilon - \gamma_2) U''(q_\varepsilon)} > 0, \quad \text{for } \varepsilon \in (1, \hat{\varepsilon}]. \quad (57)$$

Consequently, inflation not only curtails consumption due to lack of liquidity for those buyers with a great desire to consume ($\varepsilon > \hat{\varepsilon}$), but it also increases consumption for those buyers with a low appetite for goods ($\varepsilon < \hat{\varepsilon}$).

As in the full information model, the inefficiencies described in the previous paragraph arise only with positive nominal interest rates. If $i \rightarrow 0$ (Friedman Rule), the cash constraint never binds: $\hat{\varepsilon} = \bar{\varepsilon}$. Also, $\gamma_1 = 1$ and $\gamma_2 = 0$, so the quantities traded are efficient because they obey: $U'(q) = C'(q)$.

6 Conclusion

Our analysis shows that a precise modeling of private information in a monetary competitive search environment brings interesting new insights about the effect of inflation. In particular, it constructs a rigorous model where the primary effect of inflation is to obstruct role of the price system in achieving a correct allocation of goods. In equilibrium, individuals sometimes purchase goods for which they care little; while some other times, they lack the liquidity to purchase the goods for which they care a lot. The intuition of this outcome is the following. Inflation gives buyers an incentive to reduce their money balances. Sellers, aware of this incentive, compete with one another by posting price offers that reduce the precautionary money balances that buyers need to carry. With full information, sellers can do so by posting an offer which consists of a single flat fee independent of the quantities served. With private information of preference shocks, incentive compatibility constraints imply that the payments buyers make must increase with the quantities they purchase, so price schedules must be upward sloping. However, inflation makes these price schedules relatively flat. Consequently, the marginal cost of purchasing goods falls with inflation, so individuals purchase inefficiently large quantities of goods as long as they are not cash constrained. Therefore, inflation ends

up shifting output from the cash constrained individuals with a high valuation for goods to individuals with low valuations.

Appendix A

Proof Proposition 3.1

Consider the problem of an individual in an equilibrium where all other individuals have value functions (24). These other individuals have initial wealth in the interval $[\underline{a}, \bar{a}]$. Throughout the appendix, we use without further proof the absence of uncertainty in trading opportunities because of efficient matching.

For all finite $a \geq a_{\min}$, the set of feasible time and state contingent policies is non empty. The feasible values of the quantities consumed and produced are bounded. Also, for all the feasible policies the present discounted utility is well defined and finite because \mathcal{U} is a continuous function. Consequently, we can use standard recursive methods to find the value function.

In competitive search, we can recursively characterize the individual optimization problem as follows. (This characterization uses a more general definition of competitive search than in Section 4 because it allows the individual to have wealth outside the interval $[\underline{a}, \bar{a}]$.) The individual first chooses whether to be a buyer or a seller. As a seller, the individual chooses $\left(\{q_\varepsilon^s, z_\varepsilon^s\}_{\varepsilon \in [1, \bar{\varepsilon}]}, m^s, b^s\right)$, where $\{q_\varepsilon^s, z_\varepsilon^s\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ is the offer posted. As a buyer, the individual chooses $\left(\{q_\varepsilon^b, z_\varepsilon^b, \mu_\varepsilon^b\}_{\varepsilon \in [1, \bar{\varepsilon}]}, m^b, b^b\right)$, where $\{q_\varepsilon^b, z_\varepsilon^b\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ is the offer in the submarket where the buyer chooses to trade. These choices are subject to the constraints (14)-(17), (19)-(22), and (23). Moreover, in the financial markets the individual takes as given the rate of interest and the insurance premia. In the goods market, the individual takes as given the reservation expected trade surpluses of other traders and has rational expectations about their actions. Therefore, as a seller, the individual posts an offer which guarantees buyers their reservation expected trade surplus: $\int_1^{\bar{\varepsilon}} [\varepsilon U(q_\varepsilon^s) - z_\varepsilon^s] dF(\varepsilon) - i \max\{z_\varepsilon^s\}_{\varepsilon \in [1, \bar{\varepsilon}]} \geq \bar{S}^b$. As a buyer, the individual acts as if he/she were choosing $\{q_\varepsilon^b, z_\varepsilon^b\}_{\varepsilon \in [1, \bar{\varepsilon}]}$ that satisfies $\int_1^{\bar{\varepsilon}} [z_\varepsilon^b - C(q_\varepsilon^b)] dF(\varepsilon) = \bar{S}^s$, because competition among sellers implies that all posted offers yield the same expected trade surplus \bar{S}^s to the sellers.

Let $\mathcal{C}(a)$ be the space of bonded and continuous functions $f : [a_{\min}, \infty) \rightarrow R$, with the sup norm. Use the Bellman's equations (13) and (18) together with (12) to define the mapping T of $\mathcal{C}(a)$ onto itself by substituting f for V in the right hand sides of (13) and (18) and denoting as $Tf(a)$ the left hand side of (12). The choice variables and constraints of these

maximization programs are described in the previous paragraph. For a given a , the set of feasible policies is non-empty, compact-valued, and continuous. The utility function \mathcal{U} is a bounded and continuous on the set of feasible policies, and $0 < \beta < 1$. Therefore, Theorem 4.6 in Stokey and Lucas with Prescott (1989) implies that there is a unique fixed point to the mapping T , which is the value function V .

Let $\mathcal{V}(a)$ be the set of functions $f : [a_{\min}, \infty) \rightarrow R$ that satisfy (24) where v_0 , \bar{a} , and \underline{a} are given by

$$\begin{aligned} v_0 &= \frac{\bar{S}^s}{1-\beta} + \frac{\beta}{1-\beta} \frac{\gamma-1}{\gamma}, \\ \bar{a} &= \frac{\int_1^{\bar{\varepsilon}} z_{\varepsilon} dF(\varepsilon) + im}{1-\beta} - \frac{\beta}{1-\beta} \frac{\gamma-1}{\gamma}, \text{ and} \\ \underline{a} &= -\frac{\int_1^{\bar{\varepsilon}} z_{\varepsilon} dF(\varepsilon)}{1-\beta} - \frac{\beta}{1-\beta} \frac{\gamma-1}{\gamma}; \end{aligned} \tag{58}$$

where i , m , \bar{S}^s , and z_{ε} satisfy the equilibrium system of equations described in 4. Consider the mapping T defined in the previous paragraph. Since V is concave, it is an optimal policy to fully insure preference shocks (full insurance is strictly optimal if there is a positive probability that $a_{+1} \notin [\underline{a}, \bar{a}]$). In consequence, a_{+1} is not stochastic. Let a_{+1}^b be next period real wealth for an optimal policy conditional on being a buyer. Similarly, let a_{+1}^s be the optimal policy conditional on being a seller. If $a_{+1}^b, a_{+1}^s \in [\underline{a}, \bar{a}]$, $TV(a)$ is the maximum of $V^b(a)$ and $V^s(a)$ in equations (25) and (26), so $TV(a)$ is affine and the trade surpluses are those in (27) and (28). The optimal policies of the individual are the equilibrium ones characterized in Section 4. Therefore, the individual is indifferent between being a buyer or a seller. This indifference is broken when one policy would lead to $a_{+1} \notin [\underline{a}, \bar{a}]$. In such a case, the strict concavity of V outside the interval $[\underline{a}, \bar{a}]$ implies that it is suboptimal to be a seller if $a_{+1}^s > \bar{a}$. Likewise, it is suboptimal to be a buyer if $a_{+1}^b < \underline{a}$. Consequently, the recursive constraints (14) to (16) and (19) to (21), together with (58), imply that $a_{+1} \in [\underline{a}, \bar{a}]$ if and only if $a \in [\underline{a}, \bar{a}]$. This implies that $TV(a)$ is affine in the interval $[\underline{a}, \bar{a}]$. Equation (26) implies that the constant term of this affine function is the value of v_0 in (58). If $a > \bar{a}$, the optimal policy is to be a buyer. Vice versa, if $a < \underline{a}$, an optimal policy is to be a seller. In both cases, the strict concavity of U and convexity of C imply the strict concavity of $TV(a)$ for $a \notin [\underline{a}, \bar{a}]$. In summary, T maps $\mathcal{V}(a)$ onto itself. Therefore, the value function V satisfies

(24). Finally, since V is concave, \mathcal{U} is continuously differentiable and the solution is interior, V is continuously differentiable.

Appendix B

Competitive Search Equilibrium with Private Information

In this section, we solve program (43) to (48) in two stages. Stage 1 (Statements 1 to 13) solves for the program for a given the Lagrange multiplier λ associated with constraint (44), and given m and v_1 . Stage 2 (Statements 14 to 18) endogeneizes λ , m , and v_1 .

1. Suppose $\lambda > 1/2$ and $m > -v_1$. The terms of trade in a competitive search equilibrium with private information solve the following program:¹¹

$$J(\lambda, v_1, m) = \max_{\{q_\varepsilon, v_\varepsilon\}_{\varepsilon=1}^{\bar{\varepsilon}}} \int_1^{\bar{\varepsilon}} \{v_\varepsilon + \lambda [\varepsilon U(q_\varepsilon) - C(q_\varepsilon) - v_\varepsilon]\} dF(\varepsilon) \quad (59)$$

subject to

$$\dot{v}_\varepsilon = U(q_\varepsilon), \quad (60)$$

$$z_\varepsilon \equiv \varepsilon U(q_\varepsilon) - v_\varepsilon \leq m, \quad (61)$$

$$q_\varepsilon \geq 0, \quad (62)$$

$$v_1 \text{ given.} \quad (63)$$

2. Program (59) to (63) is a standard optimal control problem where q_ε is the control variable and v_ε is the state variable. A solution to the program exists because the set of feasible paths is non-empty, bounded, and there exists a feasible path for which the objective in (59) is finite. For example, the path $q_\varepsilon = q_1$ for all ε and $v_\varepsilon = v_1 + (\varepsilon - 1)U(q_1)$, where q_1 satisfies $U_1(q_1) = v_1$, is feasible and with this path the objective in (59) is finite.
3. Suppose there is an interval $[a, b] \subseteq [1, \bar{\varepsilon}]$ of values of ε where the inequality constraint (62) is binding, that is $q_\varepsilon = 0$ for $\varepsilon \in [a, b]$. Then (60), (61), and $U(0) = 0$ imply that in this interval z_ε is constant and equal to $-v_a \leq -v_1$. Since $a \leq \bar{\varepsilon}$ and $m > -v_1$,

¹¹The constraint (47) that guarantees that q_ε is a non-decreasing function of ε is omitted for the time being because as we shall see it is not binding.

constraint (61) is not binding in $[a, b]$. Therefore, constraints (61) and (62) never bind simultaneously.

4. Suppose there is an interval $[a, b] \subseteq [1, \bar{\varepsilon}]$ of values of ε where the inequality constraint (61) is binding, that is $z_\varepsilon = m$ for $\varepsilon \in [a, b]$. Then Statement 3 implies that in this interval $q_\varepsilon > 0$, so $U(q_\varepsilon) > 0$. Hence, (60) and (61) imply that q_ε is constant in the interval $[a, b]$.
5. Let ϖ_ε denote the co-state variable associated with (60), and ς_ε and ϑ_ε be the Lagrange multipliers associated with (61) and (62) respectively. The Hamiltonian of the program (59) to (63) is:

$$\mathcal{H} = v_\varepsilon \varphi + \lambda [\varepsilon U(q_\varepsilon) - C(q_\varepsilon) - v_\varepsilon] \varphi + \varpi_\varepsilon U(q_\varepsilon) + \varsigma_\varepsilon [m - \varepsilon U(q_\varepsilon) + v_\varepsilon] + \vartheta_\varepsilon q_\varepsilon. \quad (64)$$

6. For the values of ε such that (61) is not binding, the Hamiltonian (64) is strictly concave with respect to q_ε (for these values $\varsigma_\varepsilon = 0$) and linear (and so concave) with respect to v_ε . For the values of ε such that (61) is binding, q_ε is a constant (Statement 4). Therefore, the solution to the program (59) to (63) is unique, it is characterized by the first order conditions that result from applying the Maximum Principle, and both q_ε and v_ε are continuous functions of ε .
7. The first order condition with respect to the control variable q_ε is ($\mathcal{H}_{q_\varepsilon} = 0$):

$$(\lambda \varphi - \varsigma_\varepsilon) \varepsilon U'(q_\varepsilon) + \varpi_\varepsilon U'(q_\varepsilon) = \lambda \varphi C'(q_\varepsilon) - \vartheta_\varepsilon. \quad (65)$$

The co-state variable must obey ($\mathcal{H}_{v_\varepsilon} = -\dot{\varpi}_\varepsilon$):

$$\dot{\varpi}_\varepsilon = (\lambda - 1) \varphi - \varsigma_\varepsilon. \quad (66)$$

Finally, the transversality condition implies:¹²

$$\varpi_{\bar{\varepsilon}} = 0. \quad (67)$$

¹²The transversality condition is $\varpi_{\bar{\varepsilon}} v_{\bar{\varepsilon}} = 0$. However, $v_{\bar{\varepsilon}} > 0$ if $v_1 > 0$ given $U(\cdot) \geq 0$ and (60). If $v_1 = 0$ still $v_{\bar{\varepsilon}} > 0$. If $v_{\bar{\varepsilon}} = 0$ then $v_\varepsilon = 0$ for all ε (as v_ε is non-decreasing). But this is impossible since the buyer's expected utility is strictly positive in equilibrium.

Integrating (66) for an interval $[\varepsilon, \bar{\varepsilon}]$ and using (67), the value of the co-state variable ϖ_ε is solved to obtain:

$$\varpi_\varepsilon = (\lambda - 1) \varphi(\varepsilon - \bar{\varepsilon}) + \Sigma_\varepsilon, \quad (68)$$

where, to simplify the algebraic notation, we use the following definition:

$$\Sigma_\varepsilon \equiv \int_\varepsilon^{\bar{\varepsilon}} \varsigma_u du. \quad (69)$$

Using (68), the first order condition (65) is transformed into:

$$[(2\lambda - 1) \varphi - \varsigma_\varepsilon] \varepsilon U'(q_\varepsilon) = [(\lambda - 1) \varphi \bar{\varepsilon} - \Sigma_\varepsilon] U'(q_\varepsilon) + \lambda \varphi C'(q_\varepsilon) - \vartheta_\varepsilon. \quad (70)$$

8. Suppose there is an interval $[a, b] \subseteq [1, \bar{\varepsilon}]$ of values of ε where the two inequality constraints (61) and (62) are not binding. Then the Kuhn-Tucker Theorem implies $\varsigma_\varepsilon = \vartheta_\varepsilon = 0$ for $\varepsilon \in [a, b]$, so the first order condition (70) simplifies into

$$(\varepsilon - \gamma_2) U'(q_\varepsilon) = \gamma_1 C'(q_\varepsilon) \quad \text{for } \varepsilon \in [a, b], \quad (71)$$

where

$$\gamma_1 = \frac{\lambda}{2\lambda - 1}, \text{ and } \gamma_2 = \frac{(\lambda - 1) \bar{\varepsilon} - \Sigma_b \varphi^{-1}}{2\lambda - 1}. \quad (72)$$

Since both $U'(q_\varepsilon)$ and $C'(q_\varepsilon)$ are strictly positive for q_ε strictly positive and $\lambda > 1/2$, (71) can only hold for $\varepsilon > \gamma_2$. The Implicit Function Theorem applied to (71) implies that q_ε is an increasing function of ε in the interval $[a, b]$. This property combined with (60), (61) and $U'(q_\varepsilon) \geq 0$ implies that z_ε is also increasing in the interval $[a, b]$.

9. Combining Statements 3, 4, 6, and 8, z_ε is a non-decreasing continuous function for all $\varepsilon \in [1, \bar{\varepsilon}]$. Therefore, either (61) is never binding, or it is binding in an interval of high values of $\varepsilon : [\hat{\varepsilon}, \bar{\varepsilon}]$. In such an interval, Statement 4 implies that q_ε is positive and constant: $q_\varepsilon = \hat{q}$ for $\varepsilon \in [\hat{\varepsilon}, \bar{\varepsilon}]$.
10. Combining Statements 3, 6, 8, and 9, q_ε is a non-decreasing continuous function for all $\varepsilon \in [1, \bar{\varepsilon}]$. Therefore, either (62) is never binding, or it is binding in an interval of low values of $\varepsilon : [1, \varepsilon_0]$.

11. Statements 7 to 10 imply the following characterization of the optimal path of the control variable:

$$\begin{aligned} q_\varepsilon &= 0 \quad \text{for } \varepsilon \in [1, \varepsilon_0) \text{ if } \varepsilon_0 > 1, \\ (\varepsilon - \gamma_2) U'(q_\varepsilon) &= \gamma_1 C'(q_\varepsilon) \quad \text{for } \varepsilon \in [\varepsilon_0, \hat{\varepsilon}], \text{ and} \\ q_\varepsilon &= \hat{q} \quad \text{for } \varepsilon \in [\hat{\varepsilon}, \bar{\varepsilon}] \text{ if } \hat{\varepsilon} < \bar{\varepsilon}; \end{aligned} \quad (73)$$

where

$$\gamma_1 = \frac{\lambda}{2\lambda - 1}, \text{ and } \gamma_2 = \frac{(\lambda - 1)\bar{\varepsilon} - \Sigma_{\hat{\varepsilon}}\varphi^{-1}}{2\lambda - 1}. \quad (74)$$

The two real numbers ε_0 and $\hat{\varepsilon}$ obey: $1 \leq \varepsilon_0 \leq \hat{\varepsilon} \leq \bar{\varepsilon}$.

12. If $\hat{\varepsilon} = \bar{\varepsilon}$ (condition (61) is never binding), then $\Sigma_{\hat{\varepsilon}} = 0$. If $\hat{\varepsilon} < \bar{\varepsilon}$, the first order condition (70) can be simplified using (73) and (74) for $\hat{\varepsilon}$, to obtain

$$\varsigma_\varepsilon \varepsilon = (2\lambda - 1) \varphi(\varepsilon - \hat{\varepsilon}) + \Sigma_\varepsilon - \Sigma_{\hat{\varepsilon}}. \quad (75)$$

Since $\varsigma_\varepsilon = -\dot{\Sigma}_\varepsilon$, (75) is a differential equation. Its general solution is:

$$\varsigma_\varepsilon = \frac{1}{2} (2\lambda - 1) \varphi + \frac{K}{\varepsilon^2}, \text{ and} \quad (76)$$

$$\Sigma_\varepsilon = \Sigma_{\hat{\varepsilon}} - \frac{1}{2} (2\lambda - 1) \varphi(\varepsilon - 2\hat{\varepsilon}) + \frac{K}{\varepsilon}. \quad (77)$$

The constant of integration K can be determined using the condition $\varsigma_{\hat{\varepsilon}} = 0$, so

$$K = -\frac{1}{2} (2\lambda - 1) \varphi \hat{\varepsilon}^2. \quad (78)$$

Also, the definition (69) implies $\Sigma_{\bar{\varepsilon}} = 0$. Therefore,

$$\Sigma_{\hat{\varepsilon}} = \frac{\varphi \bar{\varepsilon}}{2} (2\lambda - 1) \left[1 - 2 \frac{\hat{\varepsilon}}{\bar{\varepsilon}} + \left(\frac{\hat{\varepsilon}}{\bar{\varepsilon}} \right)^2 \right]. \quad (79)$$

Combining (79) and (74), we obtain:

$$\gamma_1 + \frac{\gamma_2}{\bar{\varepsilon}} = \frac{\hat{\varepsilon}}{\bar{\varepsilon}} + \frac{1}{2} \left[1 - \left(\frac{\hat{\varepsilon}}{\bar{\varepsilon}} \right)^2 \right]. \quad (80)$$

13. Conditional on ε_0 and $\hat{\varepsilon}$, the set of equations (73), (74), and (80) characterize the optimal path of the control variable $\{q_\varepsilon\}_{\varepsilon=1}^{\bar{\varepsilon}}$. The optimal path $\{v_\varepsilon\}_{\varepsilon=1}^{\bar{\varepsilon}}$ is obtained

from (60) and (63). If interior, the optimal values of ε_0 and $\hat{\varepsilon}$ are obtained combining the interior first order condition (71) with the constraints (62) and (61) respectively. If $\varepsilon_0 = 1$ and/or $\hat{\varepsilon} = \bar{\varepsilon}$, the constraints (62) and (61) are satisfied together with the associated Kuhn-Tucker complementary conditions.

14. The equilibrium values for λ , m , and v_1 solve the following program:

$$\max_{\{m, v_1, \lambda\}} J(\lambda, m, v_1) - im \quad (81)$$

subject to (44).

15. Since λ is the Lagrange multiplier associated with constraint (44). The first order interior conditions of program (81) can be written as follows:

$$i = J_m(\lambda, m, v_1), \text{ and} \quad (82)$$

$$J_{v_1}(\lambda, m, v_1) = 0; \quad (83)$$

together with the constraint (44).

16. Using the Envelope Theorem, (64), (69), and $\varphi = (\bar{\varepsilon} - 1)^{-1}$, conditions (82) and (83) are transformed into:

$$i = \Sigma_{\hat{\varepsilon}} \quad (84)$$

$$1 - \lambda + \Sigma_{\hat{\varepsilon}} = 0. \quad (85)$$

Therefore,

$$\lambda = 1 + i. \quad (86)$$

Conditions (84) and (86) combined with (72) and $\varphi = (\bar{\varepsilon} - 1)^{-1}$ imply

$$\gamma_1 = \frac{1+i}{1+2i}, \text{ and } \gamma_2 = \frac{i}{1+2i}. \quad (87)$$

For $i > 0$, (84) implies $\Sigma_{\hat{\varepsilon}} > 0$, so constraint (61) binds. Given γ_1 and γ_2 , the value of $\hat{\varepsilon}$ is obtained from (80).

17. Define q_1^* to be the solution to $U'(q_1^*) = C'(q_1^*)$. The assumptions about U and C imply $q_1^* > 0$. Substituting (87) into (73) implies that $q_\varepsilon \geq q_1 = q_1^* > 0$. Therefore, constraint (62) is never binding, that is $\varepsilon_0 = 1$. Equation (84) also implies for all $i > 0$ that (61) binds, so $\hat{\varepsilon} < \bar{\varepsilon}$.

18. In conclusion, the optimal path $\{q_\varepsilon\}_{\varepsilon=1}^{\bar{\varepsilon}}$ is characterized by (73), (80), (87), and $\varepsilon_0 = 1$. For i sufficiently small, the optimal values of λ , m and v_1 satisfy the assumptions made at the head of Statement 1 because of the following reasons. Equation (86) implies $\lambda > 1/2$. For $i = 0$, (84) implies $\Sigma_{\bar{\varepsilon}} = 0$, so constraint (61) is never binding. Continuity implies that for i sufficiently small $m > z_1 > -v_1$. In this case, the optimal value of m is $\hat{\varepsilon}U(\hat{q}) - v_{\bar{\varepsilon}}$. Finally, the equilibrium value of v_1 is determined by the condition $\bar{S}^s = \bar{S}^b$, that is $\int_1^{\bar{\varepsilon}} v_\varepsilon dF(\varepsilon) - im = \int_1^{\bar{\varepsilon}} [\varepsilon U(q_\varepsilon) - C(q_\varepsilon) - v_\varepsilon] dF(\varepsilon)$.
19. In the baseline model, the value of λ in (86) is independent of \bar{S}^s in (44) because utility is transferable (modifying v_1) at the rate 1 to $1 + i$. If the timing of shocks is such that we must impose the ex-post individual rationality constraint: $v_\varepsilon \geq 0$ for $\varepsilon \in [1, \bar{\varepsilon}]$, then transfers from buyers to sellers must be such that $v_1 \geq 0$. If this constraint is not binding, the competitive search equilibrium is the one characterized in previous statements because v_ε is non-decreasing with ε . If $v_1 \geq 0$ binds, (62) binds for a subset of types, which prefer not to purchase anything and pay nothing, so $\varepsilon_0 > 1$ and $q_\varepsilon = z_\varepsilon = v_\varepsilon = 0$ for $[1, \varepsilon_0)$. Equation (84) still holds, but (85) is now replaced by $1 - \lambda + \Sigma_{\bar{\varepsilon}} \leq 0$, which yields the complementary condition for $v_1 \geq 0$ to be binding. To find a solution, it is useful to combine the values of γ_1 and γ_2 in (72) with (84) to obtain:

$$\frac{(1 - \gamma_1)\bar{\varepsilon} - \gamma_2}{2\gamma_1 - 1} = \frac{i}{\varphi}. \quad (88)$$

Also, (73) implies $\varepsilon_0 = \gamma_2$. The optimal solution $\{q_\varepsilon\}_{\varepsilon=1}^{\bar{\varepsilon}}$ is characterized by (73), (80) and (88) together with $v_1 = 0$, and $\varepsilon_0 = \gamma_2$.

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